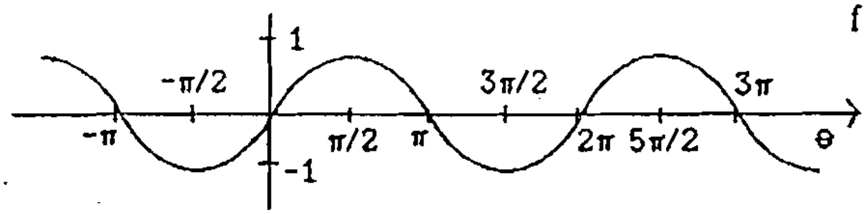
VI. Trigonometry, part 2, plus conic sections.

A. Trigonometric graphs

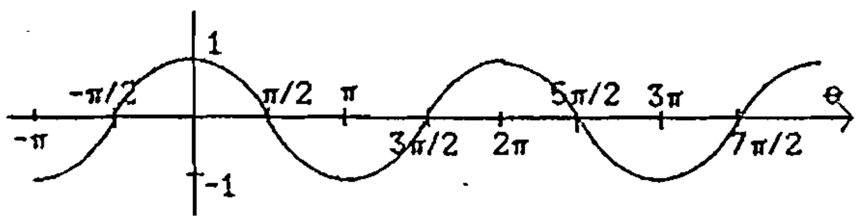
1. From definitions and data in Part 1 we can graph the trig. functions.



f(θ) = sin θ

period = 2π

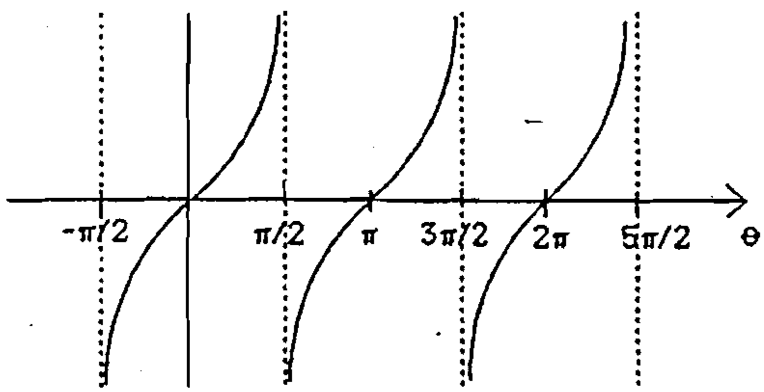
amplitude = 1



f(θ) = cos θ

period = 2π

amplitude = 1



f(θ) = tan θ

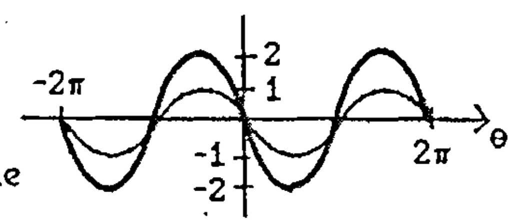
period = π

-- f(θ) = sec θ is graphed on the next page, after Example 2 --

2. Variations on the basic graphs.

Example 1

variation of amplitude



f(θ) = 2sin θ

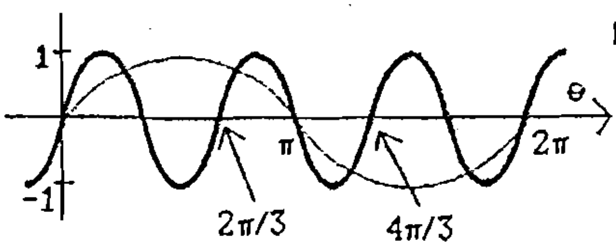
period = 2π

amplitude = 2

(sin θ shown for comparison)

Example 2

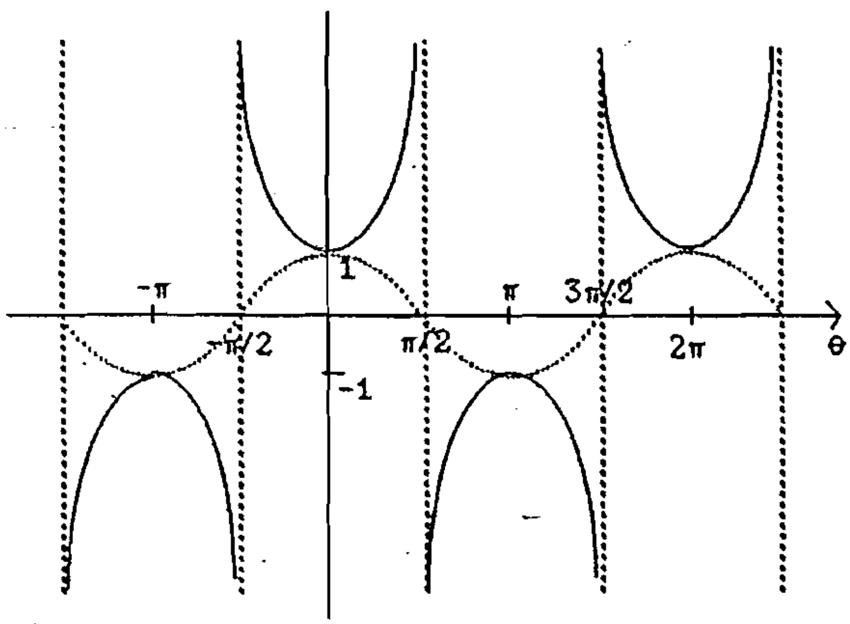
variation of period and frequency



$f(\theta) = \sin(3\theta)$

period = $2\pi/3$
amplitude = 1

As θ runs from 0 to 2π, 3θ runs from 0 to 6π. Thus the $\sin 3\theta$ curve oscillates 3 times as fast; its period is $(1/3)\cdot 2\pi$. (The graph of $\sin\theta$ is shown for comparison.)



$f(\theta) = \sec\theta$

period = 2π

(The graph of $\cos\theta$ is shown for comparison.)

B. <u>Useful identities</u> (Also see V C)

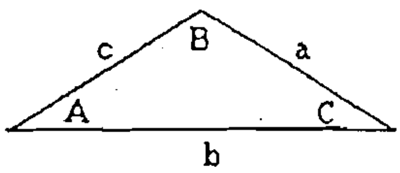
* $\sin^2\theta + \cos^2\theta = 1$ (i.e. $(\sin\theta)^2 + (\cos\theta)^2 = 1$)
$\tan^2\theta + 1 = \sec^2\theta$; $1 + \cot^2\theta = \csc^2\theta$
} Pythagorean theorem

* $\sin(\theta \pm \varsigma) = \sin\theta\cos\varsigma \pm \cos\theta\sin\varsigma$
* $\cos(\theta \pm \varsigma) = \cos\theta\cos\varsigma \mp \sin\theta\sin\varsigma$
$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$; $\cos 2\theta = 1 - 2\sin^2\theta$; $\cos 2\theta = 2\cos^2\theta - 1$

$\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$; $\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$

$\cos\theta = \sin(\theta + \pi/2)$
$\sin\theta = \cos(\theta - \pi/2)$
} Look at the graphs.

$\cos\theta = \sin(\pi/2 - \theta)$
$\sin\theta = \cos(\pi/2 - \theta)$ } Think of right triangles.



Law of sines: $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos A$
(also $b^2 = a^2 + c^2 - 2ac\cos B$, etc.)

Examples

1. Verify that for all θ, $\dfrac{\cot\theta}{\csc\theta - 1} = \dfrac{1 + \sin\theta}{\cos\theta}$.

Sol'n: multiply left side by $\dfrac{\sin\theta}{\sin\theta}$ to get $\dfrac{\cos\theta}{1 - \sin\theta}$,

then by $\dfrac{1 + \sin\theta}{1 + \sin\theta}$ to get $\dfrac{(1 + \sin\theta)\cos\theta}{1 - \sin^2\theta}$

$$= \frac{(1 + \sin\theta)\cos\theta}{\cos^2\theta} = \text{right side}.$$

2. Find $\cos\dfrac{\pi}{12}$.

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$
$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1 + \sqrt{3}}{2\sqrt{2}}.$$

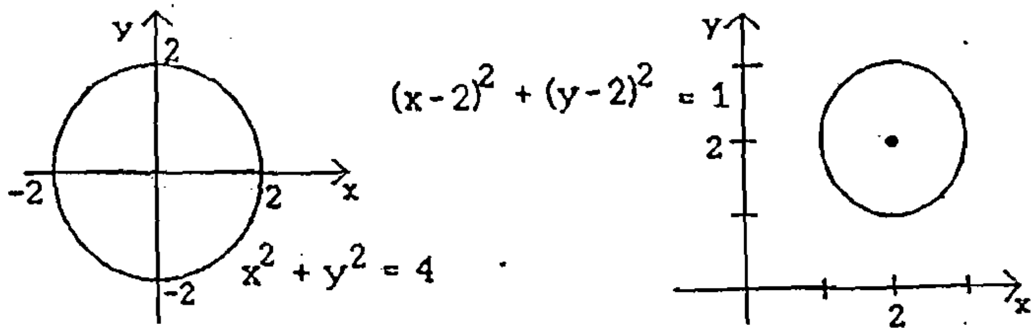
Exercises VI AB

1. Sketch the graph of $f(\theta) = -\cos\theta$.
2. Sketch the graph of $f(\theta) = 2\sin 4\theta$. What is its period?
3. Verify that $\tan\theta + \sec\theta = \dfrac{1 + \sin\theta}{\cos\theta}$.
4. Verify: $\dfrac{1 + \tan^2\theta}{\csc\theta} = \sec\theta\tan\theta$.
5. Express $\cos^2 2a$ in terms of $\cos 4a$.

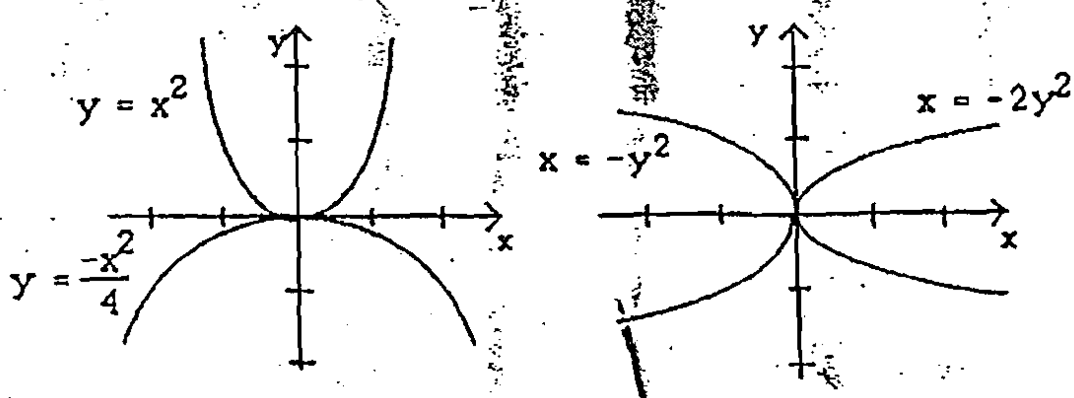
6. Find $\sin \frac{5\pi}{12}$ by using trig. identities.

C. Conic sections

1. Circles. (a) centered at origin with radius r : $x^2 + y^2 = r^2$.
   (b) centered at (h,k) with radius r : $(x-h)^2 + (y-k)^2 = r^2$.



2. Parabolas.
   (a) Vertex at origin: $y = kx^2$ or $x = ky^2$. The larger $|k|$ is, the narrower the parabola will be.

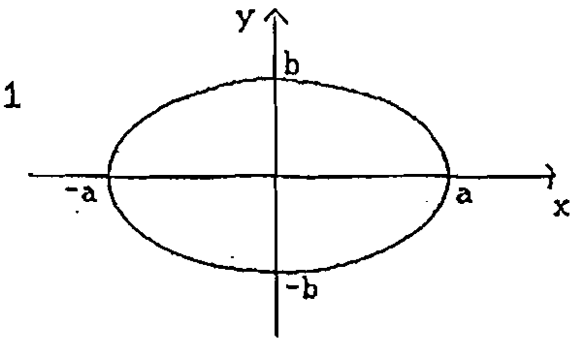


   (b) More general parabola will be $Ax^2 + Bx + Cy + D = 0$ or $Ay^2 + By + Cx + D = 0$ with $A, C \neq 0$. By completing the square, these may be rewritten as $y = a(x-b)^2 + c$ or $x = a(y-b)^2 + c$, which are parabolas with vertices at (b,c) or (c,b), respectively.

3. Ellipses.

(a) Center at origin, x-intercepts $\pm a$, y-intercepts $\pm b$, $a, b > 0$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



major semi-axis = a,
minor semi-axis = b
since $a > b$.

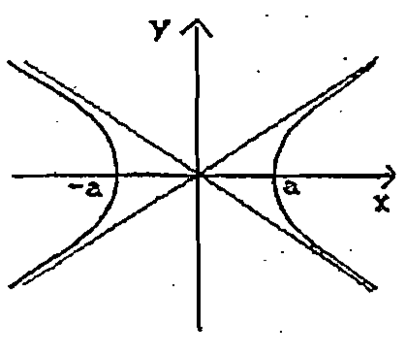
(b) More general ellipse: $Ax^2 + By^2 + Cx + Dy + E = 0$ with $A, B \neq 0$, A, B same sign. Rewrite as

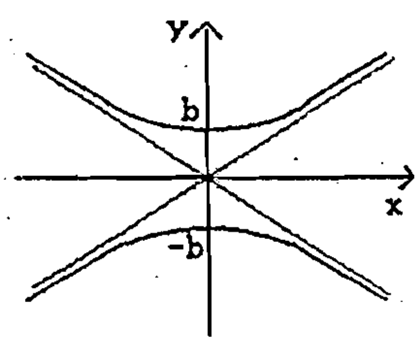
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \text{ ; center at } (h,k) \text{ , semi-axes } a \text{ and } b.$$

4. Hyperbolas.

(a) Center at origin, asymptotes $ay = \pm bx$, $a, b > 0$:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{or} \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$



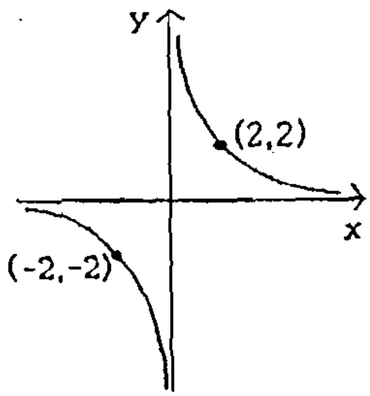


(b) More general hyperbola: $Ax^2 + By^2 + Cx + Dy + E = 0$, $A, B \neq 0$, A, B opposite signs.

Rewrite as $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = \pm 1$, center at (h,k).

(c) $xy = c$ is also a hyperbola; write as $y = x/c$ to graph; for example,

$y = \frac{4}{x}$



Note: To display the equations of conic sections in a standard, recognizable form as above, it may be necessary to "complete the square". The technique comes from the formula $(x + a)^2 = x^2 + 2ax + a^2$; we can write $x^2 + bx + c = (x + b/2)^2 + c - b^2/4$, getting rid of the x term. (In the exercises, this has already been done.)

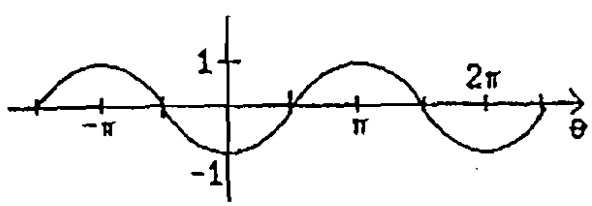
Exercises VI C

Identify and sketch the following conic sections.

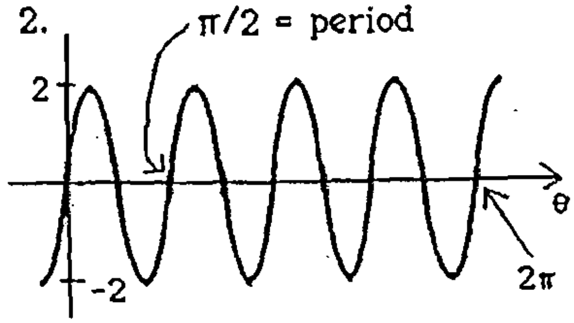
1. $8x = -3(y-(1/3))^2 + 4$
2. $(x-1)^2 + (y-2)^2 = 9$
3. $9x^2 + 4y^2 = 36$

Answers for Exercises VI

A: 1.



2.



B: 3. Hint: write the left side in terms of $\sin\theta$ and $\cos\theta$.