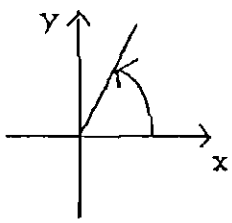
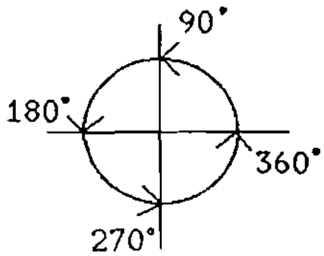
V. Trigonometry, part 1

A. Angle measurement



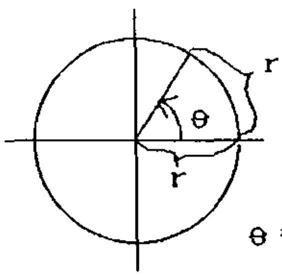
The standard position for angles in the xy-plane is with the initial side on the positive x-axis and the counter-clockwise direction taken to be positive.

Two common units for measuring angles are degrees and radians.

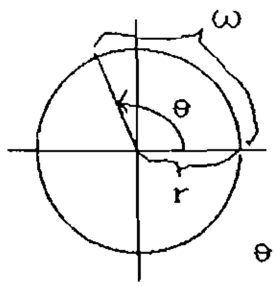


Degrees $360^\circ = 1$ revolution, so $1^\circ = \frac{1}{360}$ th of a revolution.

Radians. One radian is defined to be the angle subtended at the center by an arc of length r on a circle of radius r. The circumference of a circle of radius r has length $2\pi r$, so r units can be marked off "2π times". In other words, 1 revolution = 2π radians.



$\theta = 1$ radian



$\theta = \frac{\omega}{r}$ radians

Relation between degrees and radians:

Since 1 revolution = 360° = 2π radians, $180^\circ = \pi$ radians.
This gives the conversion relations:

To convert degrees to radians, multiply by $\dfrac{\pi \text{ radians}}{180^\circ}$,

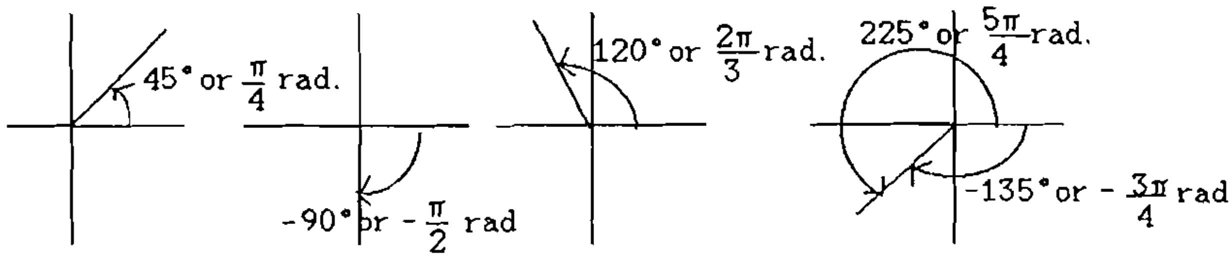
e.g. $45^\circ = 45^\circ \cdot \dfrac{\pi \text{ radians}}{180^\circ} = \dfrac{\pi}{4}$ radians.

To convert radians to degrees, multiply by $\frac{180^\circ}{\pi \text{ radians}}$,

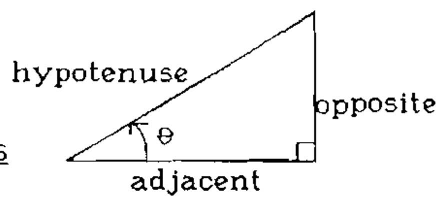
e.g. $\quad -\frac{\pi}{3}$ radians $= -\frac{\pi}{3}$ radians $\cdot \frac{180^\circ}{\pi \text{ radians}} = -60^\circ$.

In calculus, radian measure is used exclusively (because it simplifies the differentiation formulas for the trig. functions).

Examples: some angles in standard position.



B. Trig. functions for angles in right triangles



1. If θ is an acute angle (more than $0°$ but less than $90°$), the trigonometric functions of θ can be defined as ratios of sides in a right triangle having θ as one angle. The six possible ratios give the six trig. functions.

sine of θ = $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$; cosecant of θ = $\csc(\theta) = \frac{\text{hyp}}{\text{opp}}$

cosine of θ = $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$; secant of θ = $\sec(\theta) = \frac{\text{hyp}}{\text{adj}}$

tangent of θ = $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$; cotangent of θ = $\cot(\theta) = \frac{\text{adj}}{\text{opp}}$

Notes: We usually write $\sin\theta$ instead of $\sin(\theta)$, etc.
Cotangent of θ is sometimes abbreviated $\text{ctn}(\theta)$.
Remember to think "sine of θ" , <u>not</u> "sine times θ" !!

2. <u>Reciprocal relations</u> : $\csc\theta = \dfrac{1}{\sin\theta}$ and $\sin\theta = \dfrac{1}{\csc\theta}$

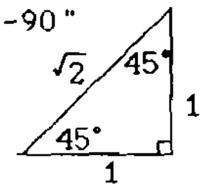
$\sec\theta = \dfrac{1}{\cos\theta}$ and $\cos\theta = \dfrac{1}{\sec\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$ and $\tan\theta = \dfrac{1}{\cot\theta}$

Also note that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

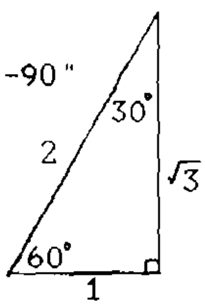
3. <u>The most common triangles</u>:

"45-45-90"



$\sin 45^\circ = \cos 45^\circ = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$; $\tan 45^\circ = 1$.

"30-60-90"

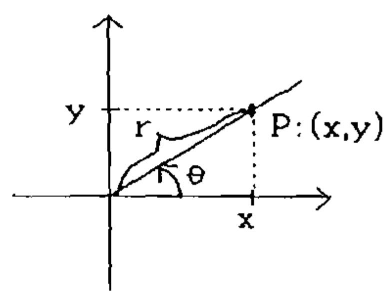


$\sin 60^\circ = \cos 30^\circ = \dfrac{\sqrt{3}}{2}$

$\sin 30^\circ = \cos 60^\circ = \dfrac{1}{2}$

4. For other angles in right triangles, consult trig. tables or use a calculator. (Take care, however -- be sure you and your calculator agree on whether you're punching in degree measure or radian measure!)

C. <u>Trig. functions for general angles</u>



1. Place the angle θ in standard position. Mark a point $P:(x,y)$ on its terminal side. Let r be the distance from the origin to P . Then the six trig. functions of θ are defined by:

$$\sin\theta = \frac{y}{r}; \quad \cos\theta = \frac{x}{r}; \quad \tan\theta = \frac{y}{x};$$

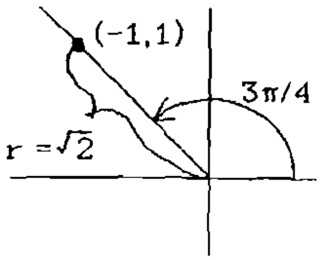
$$\csc\theta = \frac{r}{y}; \quad \sec\theta = \frac{r}{x}; \quad \cot\theta = \frac{x}{y}.$$

provided the ratio in question is defined (does not have zero in the denominator).

Notes: When θ is an acute angle, this definition gives the same results as the definition in Section B.

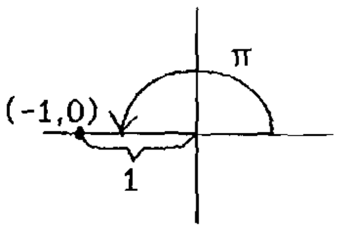
If P is chosen so that $r = 1$, the formulas simplify; $\cos\theta$ and $\sin\theta$ are the x and y coordinates of the appropriate point on the unit circle $x^2 + y^2 = 1$.

2. Examples



$$\cos\frac{3\pi}{4} = \frac{-1}{\sqrt{2}} \text{ or } \frac{-\sqrt{2}}{2} \qquad \tan\frac{3\pi}{4} = \frac{+1}{-1} = -1$$

$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$



$$\cos\pi = \frac{-1}{1} = -1 \qquad \tan\pi = \frac{0}{-1} = 0$$

$$\sin\pi = \frac{0}{1} = 0$$

but $\csc\pi$, which would be $\frac{1}{0}$, is undefined.

3. Trig. functions at the popular angles

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef'd	0	undef'd

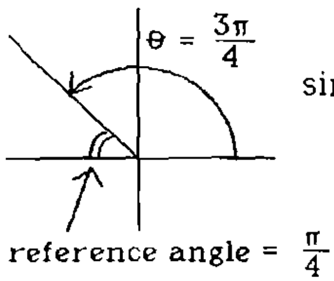
4. Reference angles. If θ is not a quadrantal angle (the terminal side is not an axis), then

sin + cos, tan −	all +
tan + sin, cos −	cos + sin, tan −

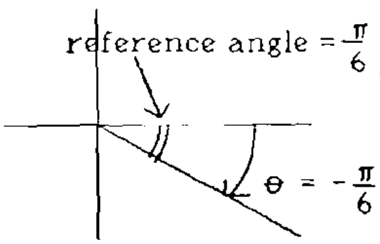
Trig. fn. of θ = ± same trig. fn. of reference angle for θ,

and the choice of ± is determined by the chart at left. The reference angle for θ is the smallest unsigned angle between the terminal side of θ and the x-axis.

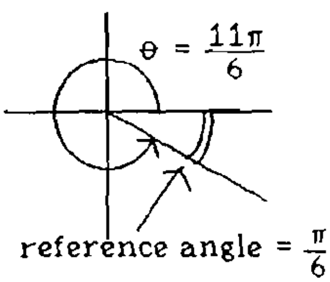
Examples:



$$\sin\frac{3\pi}{4} = +\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\ ; \quad \cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}};$$
$$\tan\frac{3\pi}{4} = -\tan\frac{\pi}{4} = -1\ .$$



$$\sin -\frac{\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}; \quad \cos -\frac{\pi}{6} = +\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2};$$
$$\tan -\frac{\pi}{6} = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}\ .$$



The trig functions of $\frac{11\pi}{6}$ are the same as those of $-\frac{\pi}{6}$, computed above.

This method is based on the following identities, which come from the definitions of the trig. functions:

$$\sin(-\theta) = -\sin\theta\ ; \qquad \cos(-\theta) = \cos\theta\ ;$$
$$\sin(\theta+\pi) = -\sin\theta\ ; \qquad \cos(\theta+\pi) = -\cos\theta\ .$$

Also, note that the trig. functions are periodic: they repeat every 2π radians: $\sin(\theta+2\pi) = \sin\theta$; $\cos(\theta+2\pi) = \cos\theta$, etc.

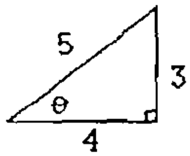
See the next handout, "Trigonometry, part 2, plus conic sections", for other trig. identities.

Exercises V ABC

1. Convert to radians. (a) $15°$ (b) $450°$
2. Convert to degrees. (a) $5\pi/6$ radians (b) $-5\pi/2$ radians
3. For each of the following angles, find $\sin\theta$, $\cos\theta$, $\tan\theta$. Use reference angles where appropriate.

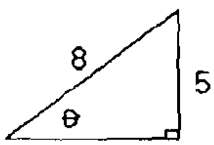
   (a) $5\pi/4$ (b) $-11\pi/6$ (c) 3π (d) $-60°$
4. Find $\csc(4\pi/3)$, $\sec(4\pi/3)$, $\cot(4\pi/3)$.

5. Given:



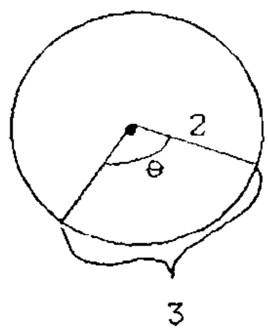
Find (a) $\sin\theta$
(b) $\cos\theta$
(c) $\tan\theta$

6. Given:

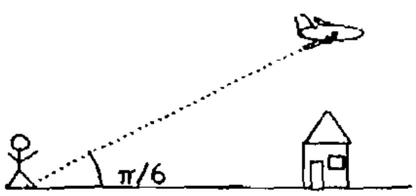


Find (a) the length of the third side
(b) $\sin\theta$
(c) $\cos\theta$
(d) $\tan\theta$

7. In the diagram at right, find the radian measure of θ .



8.



1 mi. = 5280 ft.

An airplane is flying over a house 1 mile from the spot where you're standing. At that instant, the plane's angle of elevation from your viewpoint is $\pi/6$ radians. Find the plane's altitude (in feet).

Answers to Exercises **V**

1.(a) $\pi/12$ (b) $5\pi/2$

2.(a) $150°$ (b) $-450°$

3.(a) $-1/\sqrt{2}$, $-1/\sqrt{2}$, 1 (b) $1/2$, $\sqrt{3}/2$, $1/\sqrt{3}$

(c) 0 , -1 , 0 (d) $-\sqrt{3}/2$, $1/2$, $-\sqrt{3}$

4. $-2/\sqrt{3}$, -2 , $1/\sqrt{3}$

5.(a) $3/5$ (b) $4/5$ (c) $3/4$

6.(a) $(\text{side})^2 + 5^2 = 8^2$ so $(\text{side}) = \sqrt{64-25} = \sqrt{39}$

(b) $5/8$ (c) $\sqrt{39}/8$ (d) $5/\sqrt{39}$ or $5\sqrt{39}/39$

7. By ratios: θ is to angle of whole circle as 3 is to circumference,

$$\text{or } \frac{\theta}{2\pi} = \frac{3}{2\pi \cdot 2}, \text{ so } \theta = \frac{3}{2}.$$

8. $$\tan\frac{\pi}{6} = \frac{\text{alt.}}{1\text{ mi.}}, \text{ so alt.} = \frac{\sqrt{3}}{3} \cdot 5280 \text{ ft.} = 1760\sqrt{3} \text{ ft.}$$