VII. Inequalities and absolute value

A. <u>Properties of inequalities</u> (stated for $<$, but corresponding rules hold for $>$, $\leq$, and $\geq$)

1. If $a<b$ and c is any number, then $a+c<b+c$.
2. If $a<b$ and c is <u>positive</u>, then $ac<bc$.
3. However, if $a<b$ and c is <u>negative</u>, then $ac>bc$, i.e. the inequality is reversed.
4. If $a<b$ and a and b are both positive or both negative, then $1/a>1/b$; the inequality reverses. (If they are of opposite signs, the inequality stays the same; positive > negative.)

For illustration, $3<5$ so $3(4)<5(4)$, but $3(-4)>5(-4)$ and $1/3>1/5$.
$12<20$ $\qquad$ $-12>-20$

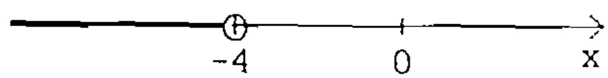
B. <u>Solving inequalities</u> Working with inequalities is much like working with equations -- but note the differences expressed by properties 3 and 4 above.

<u>Examples</u>

1. Find all real numbers x satisfying $8x+2<3x-18$.

$$8x+2<3x-18$$
$$8x-3x<-18-2$$
$$5x<-20$$
$$x<-4$$

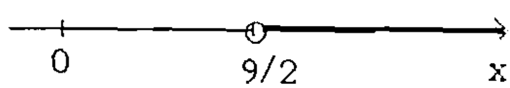
Geometrically:



2. Find all x satisfying $3x+2<5x-7$.

$$3x+2<5x-7$$
$$3x-5x<-7-2$$
$$\left.\begin{array}{r}-2x<-9\\ x>9/2\end{array}\right\}\text{ Reverse the inequality.}$$

Geometrically:

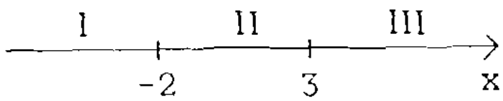


3. Find all x satisfying $x(x-1)\leq 6$.

$$x(x-1)\leq 6$$
$$x^2-x\leq 6$$
$$x^2-x-6\leq 0$$
$$(x-3)(x+2)\leq 0$$

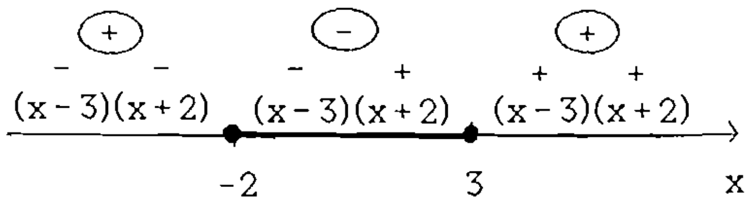
Proceed as for quadratic equations: put all terms on one side with 0 on other. Then factor.

Note that the expression on the left-hand side can change sign only where it passes through zero, namely at $x=3$ and at $x=-2$. These two points divide the x-axis into three regions:



In I, where $x < -2$, both $(x-3)$ and $(x+2)$ are negative. Therefore the product $(x-3)(x+2)$ is positive.
In II, where $-2 < x < 3$, $(x+2)$ is positive and $(x-3)$ is negative, so their product is negative.
In III, where $x > 3$, both $(x+2)$ and $(x-3)$ are positive, so their product is positive.

Summarizing:



Thus the solution is $-2 \le x \le 3$.

4. Find all x : $\dfrac{3x}{x-1} < \dfrac{x}{x+2} + 2$.

If we tried to clear fractions by multiplying by $(x-1)(x+2)$ we'd have to divide into cases according to whether $(x-1)(x+2)$ is positive or negative. It's easier to put all terms on one side and add fractions.

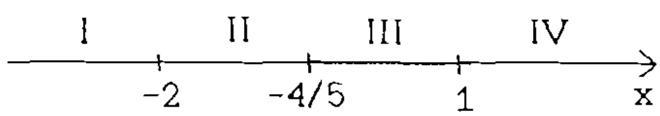
$$\frac{3x}{x-1} - \frac{x}{x+2} - 2 < 0$$

$$\frac{3x(x+2) - x(x-1) - 2(x-1)(x+2)}{(x-1)(x+2)} < 0$$

$$\frac{5x+4}{(x-1)(x+2)} < 0$$

Sign changes can occur when $5x + 4 = 0$ ..... $x = -4/5$
$x - 1 = 0$ ..... $x = 1$
$x + 2 = 0$ ..... $x = -2$.

Four regions to consider:



Solution from chart on next page: the inequality holds when $x < -2$ <u>or</u> $-4/5 < x < 1$.

	$x+2$	$5x+4$	$x-1$	$(5x+4)/(x-1)(x+2)$
$x<-2$	$-$	$-$	$-$	$-$
$-2<x<-4/5$	$+$	$-$	$-$	$+$
$-4/5<x<1$	$+$	$+$	$-$	$-$
$x>1$	$+$	$+$	$+$	$+$

Exercises **VII** B Find all x satisfying the given inequalities.

1. $4x+3>0$
2. $3x+2<4x-3$
3. $x^2-x-12<0$
4. $x(x+4)<-3$
5. $(x-1)^2(x-2)^4>0$
6. $(x-1)^3>0$
7. $\dfrac{(x-4)^3(x+2)}{(x-1)^2} \geq 0$
8. $\dfrac{x-3}{x+1} < 2$
9. $(1/3)x^{-2/5}(x-7)^{-2} + 2x^{1/3}(x-7) > 0$

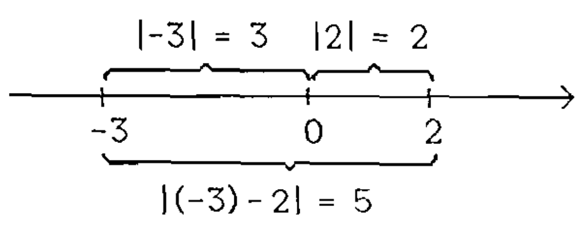
C. Absolute value

Geometrically, the absolute value of a real number a, denoted $|a|$, is the distance on a number line between 0 and a.

Algebraically, $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0, \end{cases}$

e.g. $|2| = 2$; $|-3| = -(-3) = 3$; $|0| = 0$.

Note that $|a-b|$ is the distance between a and b.

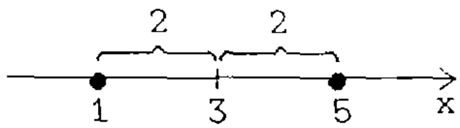


Examples 1. Find all x such that $|x-3| = 2$.

Algebraic solution: $x-3 = 2$ or $x-3 = -2$

$x = 5$ or $x = 1$

Geometric solution:
distance between x and 3 equals 2; $x = 1$ or 5.

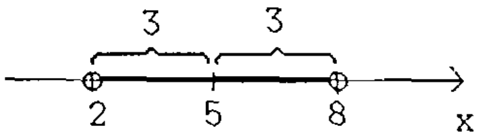


2. Find all x such that $|x-5| < 3$.

Algebraic:

$$-3 < x-5 < 3$$
$$-3+5 < x < 3+5$$
$$2 < x < 8$$

Geometric: distance between x and 5 is less than 3.



3. Find all x such that $|3x-10| > 2$.

Either $3x-10 > 2$ <u>or</u> $3x-10 < -2$

$3x > 12$ <u>or</u> $3x < 8$

$x > 4$ <u>or</u> $x < 8/3$

(Geometrically, distance between $3x$ and 10 is greater than 2.)

4. Find all x such that $\left| \dfrac{2}{1-x} \right| < 1$.

$$-1 < \frac{2}{1-x} < 1$$

Taking reciprocals reverses the inequalities:

either $\dfrac{1}{-1} > \dfrac{1-x}{2}$ <u>or</u> $\dfrac{1-x}{2} > \dfrac{1}{1}$ subtract $1/2$

$\dfrac{-3}{2} > \dfrac{-x}{2}$ <u>or</u> $\dfrac{-x}{2} > \dfrac{1}{2}$ mult. by -2

$3 < x$ <u>or</u> $x < 1$ (reverse ineq.)

<u>Useful properties</u>:

$$|a| = |-a|$$
$$|a \cdot b| = |a||b|$$
$$|a/b| = |a|/|b|$$
$$|a+b| \le |a|+|b| \quad \text{(Triangle inequality)}$$

Exercises **VII C** Find all x satisfying the given condition.

1. $|x-2| = 5$
2. $|x-2| < 5$
3. $|x-2| > 1$
4. $|2x-3| < 5$
5. $\left| \frac{1}{2x+3} \right| < 2$

Answers to Exercises **VII**

B:
1. $x > -3/4$
2. $x > 5$
3. $-3 < x < 4$
4. $-3 < x < -1$
5. all numbers except $x = 1, 2$
6. $x > 1$
7. $x \leq -2$ or $x \geq 4$
8. $x < -5$ or $x > -1$
9. $x < 0$ or $0 < x < 1$ or $x > 7$

C:
1. $x = -7$ or 3
2. $-3 < x < 7$
3. $x > 3$ or $x < 1$
4. $-1 < x < 4$
5. $x < -7/4$ or $x > -5/4$