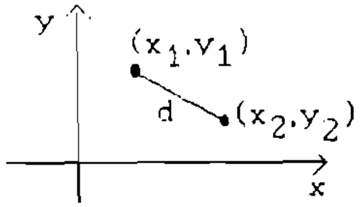
IV. Geometry and word problems

A. Geometry Review:

1. Distance in the (x,y)-plane

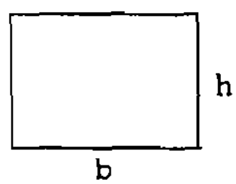


The distance d between (x_1,y_1) and (x_2,y_2)

is $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

2. Common formulas for areas and volumes

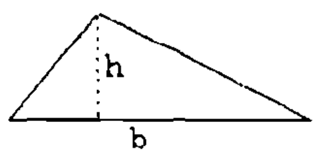
(a) Rectangle



Area = bh
(= base × height
or length × width)
Perimeter = 2b + 2h

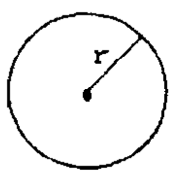
(For a square of side-length s , area = s^2 , perimeter = 4s .)

(b) Triangle



Area = ½ bh (= ½ base × height or altitude)

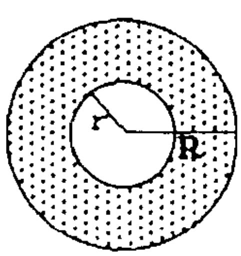
(c) Circle



Area = πr^2
Circumference = $2\pi r$

Equation of circle in the (x,y)-plane with center (h,k) and radius r : $(x-h)^2 + (y-k)^2 = r^2$.

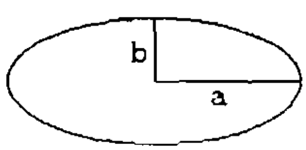
(d) Annulus



Area = (area of larger circle) - (area of smaller circle)

$= \pi R^2 - \pi r^2$

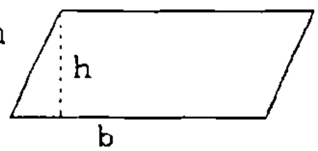
(e) Ellipse



Area = πab , where
a = major semi-axis,
b = minor semi-axis .

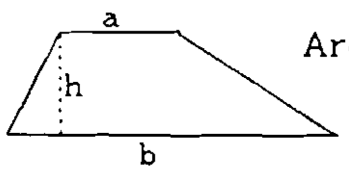
Equation of ellipse with center at $(0,0)$, x-intercepts $\pm a$ and y-intercepts $\pm b$: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(f) Parallelogram



Area = bh = base × height or altitude

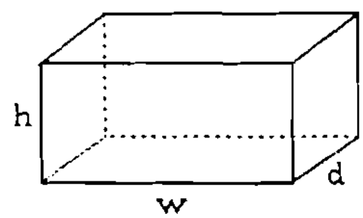
(g) Trapezoid



Area $= \frac{1}{2}h(a+b)$

$= h \cdot \dfrac{a+b}{2}$ (= height × average of the "bases")

(h) Rectangular box

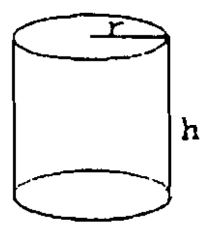


Volume = whd

Surface area = 2wd + 2dh + 2wh

For a cube with side s, volume = s^3, surface area = $6s^2$.

(i) Right circular cylinder



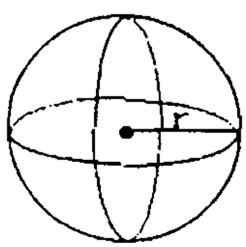
Volume $= \pi r^2 h$

= (area of circular base) × height

Surface area of curved portion (sides)

$= 2\pi r h$ = circumference × height

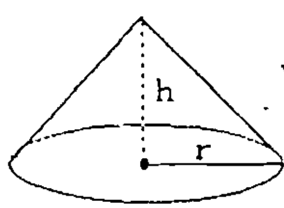
(j) Sphere



Volume $= \frac{4}{3}\pi r^3$

Surface area $= 4\pi r^2$

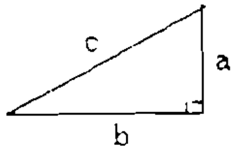
(k) Right circular cone



Volume $= \frac{1}{3}\pi r^2 h$

3. Triangle relationships

(a) Pythagorean Theorem

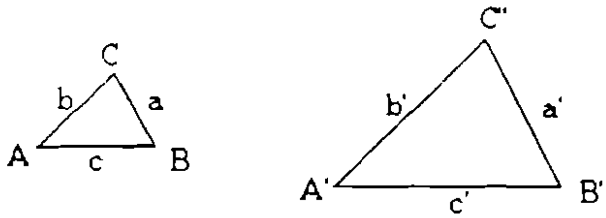


$$a^2 + b^2 = c^2$$

In a right triangle, the sum of the squares of the legs equals the square of the hypotenuse.

(b) The sum of the angles in any triangle is $180°$ or π radians.

(c) Two triangles are similar if their angles are equal. The sides of similar triangles can have different lengths but corresponding sides will be proportional.



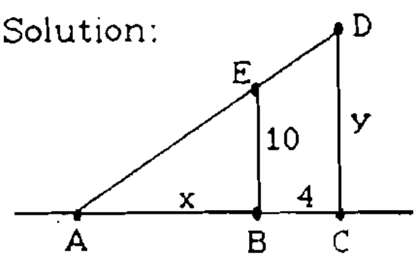
$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$$

$$\frac{a}{b} = \frac{a'}{b'}, \text{ etc.}$$

Angle at A = angle at A', etc.

Example 1 A ladder passes over a 10-foot-high fence, just touching its top, and leans against a house that is 4 feet away from the fence. Find a relationship between y, the height at the top of the ladder, and x, the distance from the foot of the ladder to the fence.

Solution:



Notice that ΔABE and ΔACD are similar. (They share angle EAB. Angle ACD and angle ABE are both right angles. The third angles, AEB and ADC, must be equal since each triangle's angles must sum to $180°$.)

Now by proportionality of corresponding sides,

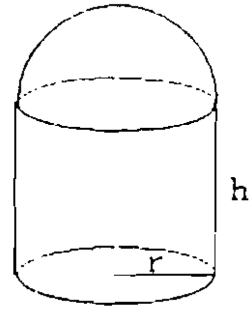
$$\frac{AC}{AB} = \frac{CD}{BE}, \quad \text{or} \quad \frac{x+4}{x} = \frac{y}{10}.$$

Example 2. A silo consists of a right circular cylinder surmounted by a hemisphere. If r and h are the radius and height of the cylinder, find a formula for the surface area of the silo.

Solution: Area = sides + roof

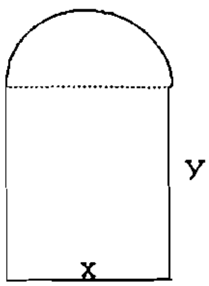
$$= 2\pi r h + \tfrac{1}{2}(4\pi r^2)$$

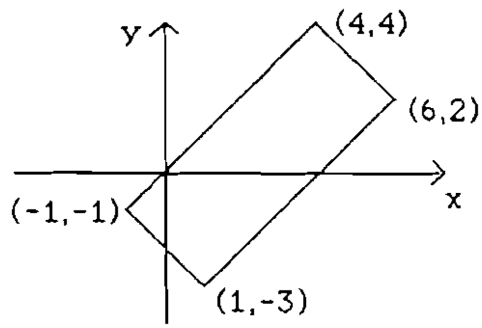
$$= 2\pi r(h+r)$$



Exercises IV A

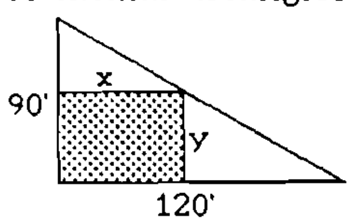
1. A Norman window consists of a rectangle surmounted by a semicircle. With x and y as shown below left, find a formula for the perimeter of the window in terms of x and y .

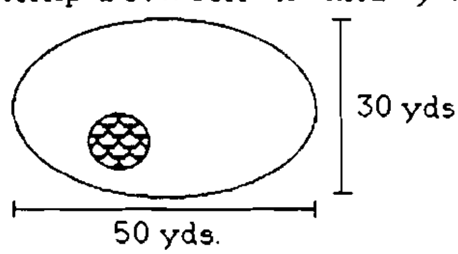




2. Find the area of the rectangle pictured above right.

3. A lot has the shape of a right triangle with legs 90 ft. and 120 ft. A rectangular building is to be built on the lot in the position shown below left. Take x and y to be the lengths of the sides of the building, as shown. Use similar triangles to find a relationship between x and y .





4. A park has the shape of an ellipse with dimensions as shown above right. It contains a circular goldfish pond 20 ft. across. What is the area of the land in the park?

B. Word problems

Many applications of mathematics involve translating words into algebra. One would like to have one method that will work for all problems; unfortunately, no completely systematic method is possible. There are, however, a few common types of problems.

1. In any problem with a geometric flavor, draw a picture and look for similar triangles, Pythagorean theorem, area formulas, etc.

2. Some sentences can be translated word-for-word into equations.

For example, " y is $\underbrace{\text{twice as large as}}_{\text{2 times}}$ $\underbrace{\text{2 less than x}}_{x-2}$ "

translates to $y = 2(x-2)$.

Look for key phrases involving proportionality:

(a) " y is (directly) proportional to x " or " y varies (directly) as x " translates to $y = kx$, where k is a constant (called the constant of proportionality) .

(b) " y is inversely proportional to x " or " y varies inversely as x " translates to $y = k/x$.

(c) " y is jointly proportional to x and z " or " y varies with x and z " translates to $y = kxz$.

<u>Example</u>

The exposure time t required to photograph an object is proportional to the square of the distance d from the object to the light source and inversely proportional to the intensity of illumination I . Express this relation algebraically.

Solution: $t = k \cdot d^2 \cdot \frac{1}{I}$ or $t = k\frac{d^2}{I}$.

3. Every term in an equation must be measured in the same units. Thus we may construct equations by forcing the units to cancel.

<u>Example</u> If a car travels 35 mph (miles per hour) for x hours and 55 mph for y hours, write an expression for D , the total distance travelled.

Solution: $\frac{\text{miles}}{\text{hour}} \cdot \text{hours} = \text{miles}$ (unit-cancelling) ,

so $\frac{35 \text{ miles}}{\text{hr}} \cdot x \text{ hrs} + \frac{55 \text{ miles}}{\text{hr}} \cdot y \text{ hrs} = D \text{ miles}$ --

$D = 35x + 55y$, all in terms of miles. (Note that "and" was translated as + .)

Exercises IV B

1. The distance d in miles that a person can see to the horizon from a point h feet above the surface of the earth is approximately proportional to the square root of the height h .

(a) Express this relation algebraically.

(b) Find the constant of proportionality if it is known that the horizon is 30 miles away viewed from a height of 400 ft.

(c) Approximately how far is the horizon viewed from a point 900 ft. high?

2. Two cars start at the same place at the same time. One car travels due north at 40 mph, the other due east at 30 mph. How far apart are the cars after t hours?

3. The Coffee Heaven Gourmet Shop sells two house blends of Brazilian and Columbian coffee beans. Each pound of Blend I contains 0.6 lbs. Brazilian and 0.4 lbs. Columbian beans, while each pound of Blend II contains 0.2 lbs. Brazilian and 0.8 lbs. Columbian. If the store wants to use up 40 lbs. Brazilian and 50 lbs. Columbian beans, how much of each blend should they make?

Answers to Exercises IV

A: 1. Perimeter $= x + 2y + \frac{1}{2}\pi x$

2. Length = dist. from $(-1,-1)$ to $(4,4)$ $= \sqrt{25+25} = 5\sqrt{2}$;
width $= \sqrt{4+4} = 2\sqrt{2}$; Area = 20 sq. units .

3. $\frac{y}{90} = \frac{120-x}{120}$ or $y = 90 - \frac{3}{4}x$.

4. Land area = ellipse - pond = $\pi \cdot 75\,\text{ft.} \cdot 45\,\text{ft.} - \pi \cdot (10\,\text{ft.})^2 = 3275\pi$ sq.ft.

B: 1.(a) $d = k\sqrt{h}$ (approx.)

(b) $d = 30$ when $h = 400$ so $30 = k\sqrt{400}$; $k = 3/2$.

(c) 45 ft.

2. $d = \sqrt{(40t)^2 + (30t)^2} = 50t$ miles .

3. Let x = lbs. Blend I , y = lbs. Blend II .
Brazilian beans: $0.6x + 0.2y = 40$ lbs.
Columbian beans: $0.4x + 0.8y = 50$ lbs.
Solve this system to get $x = 55$ lbs. , $y = 35$ lbs.